The Computational Complexity of Stochastic Galerkin and Collocation Methods for PDEs with Random Coefficients

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Stochastic Finite Element Method

Consider the stochastic elliptic problem defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $D = [0, b]^2 \subseteq \mathbb{R}^d$, $d = 2$:

$$\begin{cases} -\nabla \cdot (a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } \Omega \times D, \\ u(\omega, \mathbf{x}) = 0 & \text{on } \Omega \times \partial D, \end{cases} \tag{1}$$

with deterministic forcing $f(\mathbf{x})$ and $a(\mathbf{x}, \omega) = a(\mathbf{x}, \mathbf{y}(\omega))$, $\mathbf{y} = (y_1, \ldots, y_N) \in \Gamma$, with $\Gamma_n = y_n(\omega), \Gamma = \Gamma_1 \times \cdots \times \Gamma_N \subseteq \mathbb{R}^N$. We impose the additional assumptions on $a(\mathbf{x}, \mathbf{y})$, that

(A1) $a(\mathbf{x}, \mathbf{y}(\omega)) = a_{\min} + h(\mathbf{x}, \mathbf{y}(\omega))$ *where the y_j's are independent random variables, and $h : \mathbb{R}^N \times \mathbb{R}^d \to \mathbb{R}$.*

(A2) $a(\mathbf{x}, \mathbf{y}(\omega))$ *is almost surely uniformly bounded and coercive, i.e.* $\exists\, 0 < a_{\min} \leq a_{\max} < \infty$ *such that*

$$P(a_{\min} \leq a(\mathbf{x}, \mathbf{y}(\omega)) \leq a_{\max}) = 1, \qquad \forall \mathbf{x} \in \overline{D}$$

Also, let $\rho(\mathbf{y}) = \prod_{n=1}^{N} \rho_n(\mathbf{y})$ be the joint density of the vector $\mathbf{y}$.

The weak form of problem (1) is now given by: *find* $u \in H_0^1(D) \otimes L_\rho^2(\Gamma)$ *such that* $\forall v \in H_0^1(D) \otimes L_\rho^2(\Gamma)$

$$\int_\Gamma \int_D a(\mathbf{x},\mathbf{y}) \nabla u(\mathbf{x},\mathbf{y}) \cdot \nabla v(\mathbf{x},\mathbf{y}) d\mathbf{x} \rho(\mathbf{y}) d\mathbf{y} = \int_\Gamma \int_D f(\mathbf{x}) v(\mathbf{x},\mathbf{y}) d\mathbf{x} \rho(\mathbf{y}) d\mathbf{y}. \tag{2}$$

With some additional assumptions on the smoothness of the data $a(\mathbf{x},\mathbf{y})$ it is well known that the solution depends analytically on the parameters $y_n \in \Gamma_n$.

Stochastic Finite Element Method

Let $\{\phi_j\}_{j=1}^{J_h}$ be a finite basis for $W_h(D) \subset H_0^1(D)$, and set $J_h = \dim(W_h(D))$. We are interested in the semi-discrete approximation

$$u_{J_h}(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{J_h} u_j(\mathbf{y})\phi_j(\mathbf{x}). \tag{3}$$

given by: find $u_{J_h} \in W_h(D)$ such that

$$\int_D a(\mathbf{x}, \mathbf{y})\nabla u_{J_h}(\mathbf{x}, \mathbf{y}) \cdot \nabla v_{J_h}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_D f(\mathbf{x}) v_{J_h}(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad \rho\text{-a.e. in } \Gamma, \tag{4}$$

for all $v_{J_h} \in W_h(D)$. For any $\mathbf{y} \in \Gamma$, define

$$\boldsymbol{u}(\mathbf{y}) = [u_1(\mathbf{y}), u_2(\mathbf{y}), \ldots, u_{J_h}(\mathbf{y})].$$

Then the semi-discrete problem (4) can be written algebraically as

$$\boldsymbol{A}(\mathbf{y})\boldsymbol{u}(\mathbf{y}) = \boldsymbol{f} \quad \rho\text{-a.e. in } \Gamma, \tag{5}$$

where $\boldsymbol{A}(\mathbf{y})$ is the stochastic finite element stiffness matrix.

Stochastic Global Polynomial Subspaces

Let $p \in \mathbb{N}$ denote the polynomial order of an approximation and consider a sequence of increasing, nested multi-index sets $\Lambda(p)$ such that

$$\Lambda(0) = \{(0, \ldots, 0)\} \qquad \text{and} \qquad \Lambda(p) \subseteq \Lambda(p+1).$$

Let $\mathcal{P}_{\Lambda(p)}(\Gamma) \subset L^2_\rho(\Gamma)$ denote the multivariate polynomial space over Γ corresponding to the index set $\Lambda(p)$, defined by

$$\mathcal{P}_{\Lambda(p)}(\Gamma) = \text{span}\left\{ \prod_{n=1}^{N} y_n^{p_n} \,\middle|\, \mathbf{p} = (p_1, \ldots, p_N) \in \Lambda(p), y_n \in \Gamma_n \right\}. \tag{6}$$

We set $M_p = \dim\left\{\mathcal{P}_{\Lambda(p)}\right\}$.

Examples of $\Lambda(p)$:

- Tensor Products (TP):

$$\Lambda_{TP}(p) = \left\{ \mathbf{p} \in \mathbb{N}^N \,\middle|\, \max_n p_n \le p \right\}, \qquad M_p^{TP} = (p+1)^N$$

- Total Degree (TD):

$$\Lambda_{TD}(p) = \left\{ \mathbf{p} \in \mathbb{N}^N \,\middle|\, \sum_{n=1}^{N} p_n \le p \right\}, \qquad M_p^{TD} = (N+p)!/(N!\,p!),$$

- Hyperbolic Cross (HC):

$$\Lambda_{HC}(p) = \left\{ \mathbf{p} \in \mathbb{N}^N \,\middle|\, \sum_{n=1}^{N} \log_2(p_n+1) \le \log_2(p+1) \right\}$$

- Sparse Smolyak (SS):

$$\Lambda_{SS}(p) = \left\{ \mathbf{p} \in \mathbb{N}^N \,\middle|\, \sum_{n=1}^{N} \gamma(p_n) \le \gamma(p) \right\}, \quad \gamma(p) = \begin{cases} 0 & \text{for } p = 0 \\ 1 & \text{for } p = 1 \\ \lceil \log_2(p) \rceil & \text{for } p \ge 2. \end{cases}$$

Global stochastic Galerkin methods

Let $\{\psi_{p_n}(y_n)\}_{p=0}^{\infty}$ denote a set of $L^2_{\rho_n}$-orthonormal polynomials in Γ_n. For $\mathbf{p} \in \Lambda(p)$, we define

$$\Psi_{\mathbf{p}}(\mathbf{y}) = \prod_{n=1}^{N} \psi_{p_n}(y_n).$$

Then we see that

$$\int_{\Gamma} \Psi_{\mathbf{p}}(\mathbf{y})\Psi_{\mathbf{q}}(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \langle \Psi_{\mathbf{p}}\Psi_{\mathbf{q}} \rangle = \prod_{n=1}^{N} \langle \psi_{p_n}\psi_{q_n} \rangle = \prod_{n=1}^{N} \delta_{p_n q_n}.$$

Given the bases $\{\phi_j\}_{j=1}^{J_h} \subset W_h(D)$ and $\{\Psi_{\mathbf{p}}\}_{\mathbf{p}\in\Lambda(p)} \subset \mathcal{P}_{\Lambda(p)}(\Gamma)$, the gSGM approximation is defined by

$$u_{J_h M_p}^{gSGM}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{p}\in\Lambda(p)} u_{\mathbf{p}}(\mathbf{x})\Psi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{p}\in\Lambda(p)} \sum_{j=1}^{J_h} u_{\mathbf{p},j}\,\phi_j(\mathbf{x})\Psi_{\mathbf{p}}(\mathbf{y}). \tag{7}$$

Our goal is then to solve for the coefficients $\{u_{\mathbf{p},j}\}$, which requires the substitution of (7) into the weak formulation (2), resulting in a (possibly nonlinear) coupled system of size $J_h M_p \times J_h M_p$.

Global stochastic Galerkin methods - NISP

To deal with general nonlinear coefficients, we perform an "offline" projection of $a(\mathbf{x}, \mathbf{y})$ onto span$\{\Psi_{\mathbf{k}}(\mathbf{y})\}_{\mathbf{k}\in\Lambda(w)}$,

$$a(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} a_n(\boldsymbol{x})\Psi_n(\mathbf{y}) \approx \sum_{\mathbf{k}\in\Lambda(w)} a_{\mathbf{k}}(\boldsymbol{x})\Psi_{\mathbf{k}}(\mathbf{y}) =: \tilde{a}(\mathbf{x}, \mathbf{y}).$$

- We compute $a_{\mathbf{k}}(\mathbf{x}) = \int_\Gamma a(\mathbf{x}, \mathbf{y})\Psi_{\mathbf{k}}(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad$ for each $\mathbf{k} \in \Lambda(w)$.
- Letting $w = 2p$ yields the full Galerkin system [Matthies and Keese, 2003].
- The goal is to chose $0 \leq w \leq 2p$ such that $\|a - \tilde{a}\|_{L^2} < \text{TOL}$.

With this we can write

$$[\boldsymbol{A}(\mathbf{y})]_{i,j} \approx \sum_{\mathbf{k}\in\Lambda(w)} \Psi_{\mathbf{k}}(\mathbf{y}) \int_D a_{\mathbf{k}} \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}\mathbf{x} = \sum_{\mathbf{k}\in\Lambda(w)} \Psi_{\mathbf{k}}(\mathbf{y})[\boldsymbol{A}_{\mathbf{k}}]_{i,j} =: [\tilde{\boldsymbol{A}}(\mathbf{y})]_{i,j}$$

Global stochastic Galerkin methods - NISP

Given a sufficiently resolved stochastic finite element stiffness matrix $A(\mathbf{y}) \approx \sum_{\mathbf{k}\in\Lambda(w)} A_{\mathbf{k}}\Psi_{\mathbf{k}}(\mathbf{y}) = \tilde{A}(\mathbf{y})$, we obtain, for all $\mathbf{p} \in \Lambda(p)$,

$$\sum_{\mathbf{q}\in\Lambda(p)} \left\langle \Psi_{\mathbf{p}}(\mathbf{y}) \left(\sum_{\mathbf{k}\in\Lambda(w)} A_{\mathbf{k}}\Psi_{\mathbf{k}}(\mathbf{y}) \right) \Psi_{\mathbf{q}}(\mathbf{y}) \right\rangle u_{\mathbf{q}} = \langle f\Psi_{\mathbf{p}}(\mathbf{y})\rangle. \qquad (8)$$

By defining

$$[G_{\mathbf{k}}]_{\mathbf{p},\mathbf{q}} = \langle \Psi_{\mathbf{k}}\Psi_{\mathbf{p}}\Psi_{\mathbf{q}} \rangle \quad \text{and} \quad \tilde{K} = \sum_{\mathbf{k}\in\Lambda(w)} G_{\mathbf{k}} \otimes A_{\mathbf{k}}, \qquad (9)$$

where $G_{\mathbf{k}} \otimes A_{\mathbf{k}}$ denotes the Kronecker product of $G_{\mathbf{k}}$ and $A_{\mathbf{k}}$, we obtain the gSGM coupled system of equations, namely,

$$\tilde{K}u = F, \qquad (10)$$

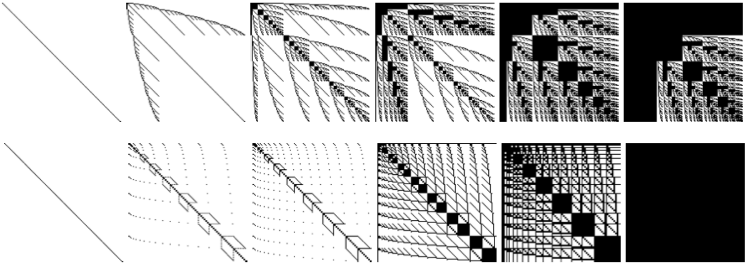
where $\tilde{K}$ approximates the full Galerkin system K when $w < 2p$.

Global stochastic Galerkin methods - Cost (solve)

Given $\tilde{\boldsymbol{K}}\boldsymbol{u} = \boldsymbol{F}$, where $\tilde{\boldsymbol{K}} = \sum_{\mathbf{k}\in\Lambda(w)} \boldsymbol{G}_{\mathbf{k}} \otimes \boldsymbol{A}_{\mathbf{k}}$, we define

$$N_G = \sum_{\mathbf{k}\in\Lambda(w)} \text{\# of nonzeros in } \boldsymbol{G}_{\mathbf{k}} = \#\left[\langle \Psi_{\mathbf{k}}\Psi_{\mathbf{p}}\Psi_{\mathbf{q}}\rangle \neq 0\right]_{\substack{\mathbf{k}\in\Lambda(w)\\ \mathbf{p},\mathbf{q}\in\Lambda(p)}}, \quad (11)$$

pictorially $N_G = \#$ of black pixels in the matrices



where each pixel represents a block matrix of the size of the original finite element system. Then N_G is the total number of nonzeros in the $\{\boldsymbol{G}_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda(w)}$.

What is N_G?

Given that the cost of solving the gSGM system depends on N_G, we must determine what is N_G and how can it be bounded.

For a total degree basis, we know that the dimension of the space is $\binom{N+p}{p}$, hence for the index $\mathbf{k} = (0, \ldots, 0)$, we have that

$$\text{nnz}(\boldsymbol{G}_{\mathbf{k}}) = \binom{N+p}{p}, \text{ since } \langle \Psi_{\mathbf{k}} \Psi_{\mathbf{p}} \Psi_{\mathbf{q}} \rangle = \langle \Psi_{\mathbf{p}} \Psi_{\mathbf{q}} \rangle.$$

Also, when $\mathbf{k} \in \Lambda(w)$ is such that $|\mathbf{k}| = 1$, we have that

$$\text{nnz}(\boldsymbol{G}_{\mathbf{k}}) \leq 2\binom{N+p-1}{p-1}$$

from a proof by [Ernst and Ullmann, 2010] in their paper on "Stochastic Galerkin Matrices".

What is N_G?

Theorem (bound on the sparsity of $G_{\mathbf{k}}$)

Given $N, p, w \in \mathbb{N}$, $0 \leq w \leq 2p$, $\mathbf{k} \in \Lambda(w)$, and even weight functions ρ_i for $i \in \{1, \ldots, N\}$, we have that

$$\text{nnz}(G_{\mathbf{k}}) = \sum_{\ell=\lceil |\mathbf{k}|/2 \rceil}^{|\mathbf{k}|} c(\mathbf{k}, |\mathbf{k}|, \ell) \binom{N+p-\ell}{p-\ell}$$

where $c(\mathbf{k}, |\mathbf{k}|, \ell)$ satisfies $c(\mathbf{k}, |\mathbf{k}|, \ell) \leq 2\binom{|\mathbf{k}|}{\ell}$, and when $\mathbf{k}$ has m ones and $N-m$ zeros (i.e. $\mathbf{k} = (1, 1, \ldots, 1, 0, \ldots, 0)$ and $|\mathbf{k}| = m$)

$$c(\mathbf{k}, |\mathbf{k}|, \ell) = \begin{cases} \binom{|\mathbf{k}|}{\ell} & \text{for } \ell = \lceil |\mathbf{k}|/2 \rceil \text{ and } |\mathbf{k}| \text{ even} \\ 2\binom{|\mathbf{k}|}{\ell} & \text{otherwise} \end{cases}$$

Examples of nnz$(G_{\mathbf{k}})$

For $N = 8$, $p = 5$

multi-index		nnz$(G_{\mathbf{k}})$	
$\lvert\mathbf{k}\rvert$	$\mathbf{k}$	actual	predicted
2	$(2,0,0,\dots,0)$	825	825
	$(1,1,0,\dots,0)$	1320	1320
3	$(3,0,0,0,\dots,0)$	420	420
	$(2,1,0,0,\dots,0)$	750	750
	$(1,1,1,0,\dots,0)$	1080	1080
4	$(4,0,0,0,0,\dots,0)$	273	273
	$(3,1,0,0,0,\dots,0)$	528	528
	$(2,2,0,0,0,\dots,0)$	693	693
	$(2,1,1,0,0,\dots,0)$	948	948
	$(1,1,1,1,0,\dots,0)$	1368	1368

What is N_G?

To count the number of matrices associated with a given total degree order, i.e. $|\mathbf{k}| = 5$, we must consider the number of ways to partition an integer to obtain multi-indices that sum to that integer.

For example:

$$\left.\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}\right\} \Rightarrow \quad q(5) = 7$$

This is a well known problem. Given $n \in \mathbb{N}$ the number of ways to sum up to n is known as the partition number $q(n)$, with generating function

$$\sum_{n=0}^{\infty} q(n) x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right).$$

What is N_G?

Finally we must consider the number of ways we can permute the indices of a given multi-index $\mathbf{k} = (k_1, \ldots, k_N)$ to count the number of associated matrices $\boldsymbol{G}_{\mathbf{k}}$ of a given type. For example, for $\mathbf{k}_1 = (4, 2, 0, 0, 0)$ and $\mathbf{k}_2 = (4, 0, 2, 0, 0)$

$$\text{nnz}(\boldsymbol{G}_{\mathbf{k}_1}) = \text{nnz}(\boldsymbol{G}_{\mathbf{k}_2}).$$

This is also a familiar counting problem:

Given the string $(4, 2, 0, 0, 0)$, how many ways are there to permute the elements when the numbers of a given type are indistinguishable.

Here switching the order of the 0's does not matter.

What is N_G?

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$$\text{nnz}(\boldsymbol{G}_{\mathbf{k}_1}) = \text{nnz}(\boldsymbol{G}_{\mathbf{k}_2}).$$

This is also a familiar counting problem:

If $n_k =$# of k's in the given string, then the answer is

$$\binom{N}{n_4, n_2, n_0} := \frac{N!}{n_4! n_2! n_0!}$$

What is N_G?

Finally we must consider the number of ways we can permute the indices of a given multi-index $\mathbf{k} = (k_1, \dots, k_N)$ to count the number of associated matrices $\boldsymbol{G}_{\mathbf{k}}$ of a given type. For example, for $\mathbf{k}_1 = (4,2,0,0,0)$ and $\mathbf{k}_2 = (4,0,2,0,0)$

$$\mathrm{nnz}(\boldsymbol{G}_{\mathbf{k}_1}) = \mathrm{nnz}(\boldsymbol{G}_{\mathbf{k}_2}).$$

This is also a familiar counting problem:

So to count the permutations of $\mathbf{k} = (4,2,0,0,0)$ we have

$$\binom{5}{1,1,3} := \frac{5!}{1!1!3!} = \frac{120}{6} = 20$$

gSGM - Cost (solve)

Since each CG iteration requires multiplying N_G matrices of the size of the finite element stiffness matrix A, the cost of solving the gSGM method with CG without preconditioning is then given by

$$W_{solve}^{gSGM} \approx N_G * N_{iter}, \tag{12}$$

where N_{iter} is the number of iterations of the system (10) required to converge to a given tolerance in CG. With preconditioning for a block diagonal Jacobi preconditioner, this becomes

$$W_{solve}^{gSGM} \approx (N_G + M_p) * N_{iter}. \tag{13}$$

Now the basic unit of cost is in terms of finite element matrix vector products, which require $\mathcal{O}(J_h)$ operations.

Global stochastic Collocation methods

1. Choose a set of points $\mathcal{H}_{M_L} = \{\mathbf{y}_k \in \Gamma\}_{k=1}^{M_L}$ according to the measure $\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \prod_{n=1}^{N} \rho_n(y_n)\,\mathrm{d}y_n$

2. For each k solve the FE solution $u_{J_h}(\mathbf{x}, \mathbf{y}_k)$ given $a(\mathbf{x}, \mathbf{y}_k)$

3. Interpolate the sampled values:
$u_{J_h M_L}^{gSCM}(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{M_L} u_{J_h}(\mathbf{x}, \mathbf{y}_k)\mathcal{L}_k(\mathbf{y})$, yielding the fully discrete gSCM approximation $u_{J_h M_L}^{gSCM} \in W_h(D) \otimes \mathcal{P}_{\mathcal{J}(L)}(\Gamma)$, where $\mathcal{L}_k \in \mathcal{P}_{\mathcal{J}(L)}(\Gamma)$ are suitable combinations of global (Lagrange) interpolants

$$\mathbb{E}[u](\mathbf{x}) \approx \int_\Gamma u_{J_h M_L}^{gSCM}(\mathbf{x}, \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \sum_{k=1}^{M_L} u_{J_h}(\mathbf{x}, \mathbf{y}_k) \underbrace{\int_\Gamma \mathcal{L}_k(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}}_{\text{precomputed weights } w_k}$$

gSCM - Cost (solve)

Therefore, the cost of solving for gSCM is given by

$$W_{solve}^{gSCM} \approx \sum_{k=1}^{M_L} N_{iter}^{(k)},$$

where $N_{iter}^{(k)}$ is the number of iterations required by CG to solve the kth FEM solution $u_{J_h}(\mathbf{x}, \mathbf{y}_k)$. Again, the cost of solving the gSGM is

$$W_{solve}^{gSGM} \approx N_G * N_{iter}.$$

Both costs are in terms of total number of matrix vector products required to find the corresponding approximation.

gSCM - Cost (solve)

Therefore, the cost of solving for gSCM is given by

$$W_{solve}^{gSCM} \approx 2 \sum_{k=1}^{M_L} N_{iter}^{(k)},$$

where $N_{iter}^{(k)}$ is the number of iterations required by CG to solve the kth FEM solution $u_{J_h}(\mathbf{x}, \mathbf{y}_k)$. Again, the cost of solving the gSGM is

$$W_{solve}^{gSGM} \approx (N_G + M_p) * N_{iter}.$$

Both costs are in terms of total number of matrix vector products required to find the corresponding approximation.

Numerical Example

We now present some results using these methods to compare gSGM and gSCM. Recall the stochastic elliptic problem

$$\begin{cases} -\nabla \cdot (a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\omega)) = \cos(x_1)\sin(x_2) & \text{in } \Omega \times D, \\ u(\mathbf{x},\omega) = 0 & \text{on } \Omega \times \partial D, \end{cases}$$

with $D = [0,b]^2$, and random coefficient $a(\mathbf{x},\omega)$ with one-dimensional (layered) spatial dependence given by

$$\log(a_N(\mathbf{x},\mathbf{y}) - 0.5) = 1 + Y_1(\omega)\left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} + \sum_{n=2}^{N} \zeta_n \varphi_n(\mathbf{x}) Y_n(\omega), \quad (14)$$

where $Y_i \sim \mathcal{U}([-\sqrt{3},\sqrt{3}])$ i.i.d.,

$$\zeta_n := (\sqrt{\pi}L)^{1/2} \exp\left(\frac{-(\lfloor \frac{n}{2} \rfloor \pi L)^2}{8}\right), \text{ if } n > 1 \quad (15)$$

and

$$\varphi_n(\mathbf{x}) := \begin{cases} \sin\left(\frac{\lfloor \frac{n}{2} \rfloor \pi x_1}{L_p}\right), & \text{if } n \text{ even}, \\ \cos\left(\frac{\lfloor \frac{n}{2} \rfloor \pi x_1}{L_p}\right), & \text{if } n \text{ odd}. \end{cases} \quad (16)$$

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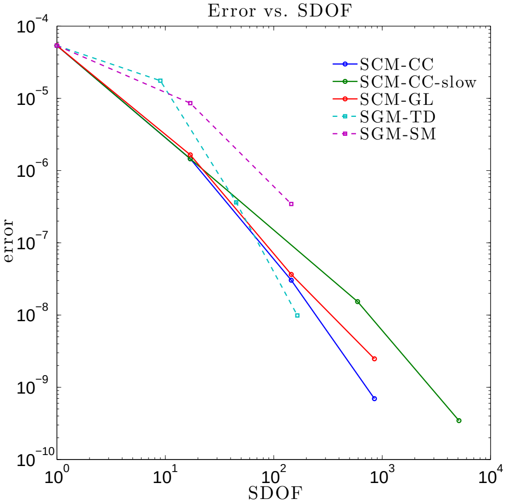
then this represents the truncation of a one-dimensional random field with stationary covariance

$$\text{Cov}[\log(a_N - 0.5)](x_1, x_2) = \exp\left(\frac{-(x_1 - x_2)^2}{L_c^2} \right),$$

and $L_c = 1/64$ is the correlation length.

Numerical Results

Here, $N = 8$ and $L_c = 1/64$ (highly isotropic).



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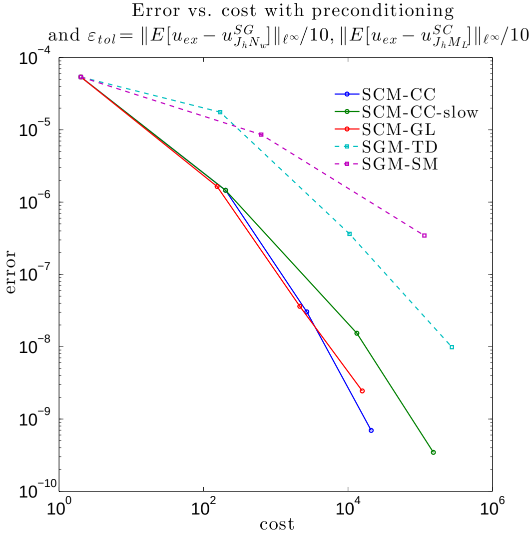

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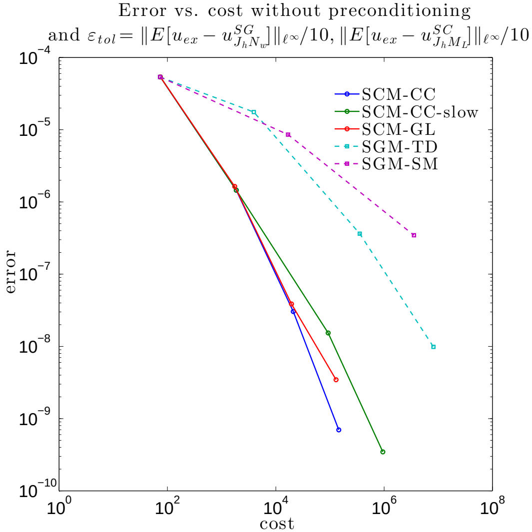


Conclusions and Future Work

- Need to compare setup cost for both methods.
- Need to obtain complexity to reach a given error estimates.

Ernst, O. G. and Ullmann, E. (2010).
Stochastic Galerkin Matrices.
SIAM Journal on Matrix Analysis and Applications, 31(4):1848–1872.

Matthies, H. G. and Keese, A. (2003).
Galerkin Methods for Linear and Nonlinear Elliptic Stochastic Partial Differential Equations.
Computer Methods in Applied Mechanics and Engineering, (July).

Extra Slides - Linear Test Case

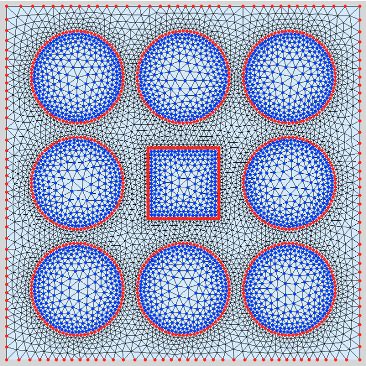
Consider the problem of isotropic thermal diffusion, that is (1) with a stochastic conductivity coefficient

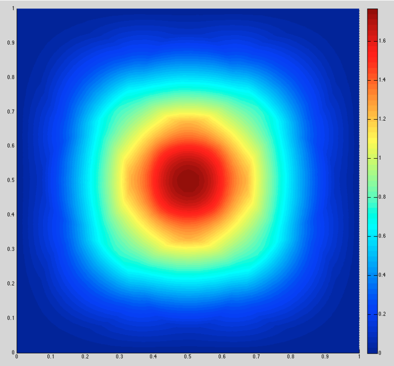
$$a(\mathbf{x}, \omega) = b_0(\mathbf{x}) + \sum_{n=1}^{8} y_n(\omega)\chi_n(\mathbf{x}),$$

with $b_0 = 1$ and $y_n(\omega) \sim \mathcal{U}(-0.99, -0.2)$, and deterministic forcing function

$$f(\mathbf{x}) = 100\chi_F(\mathbf{x}), \qquad \text{where} \qquad F = [0.4, 0.6]^2.$$

Extra Slides - Linear Test Case





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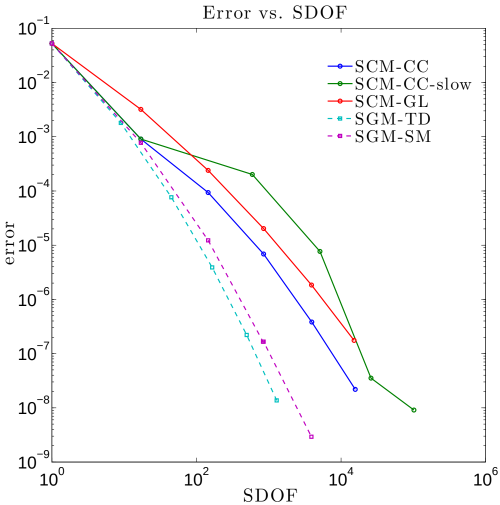

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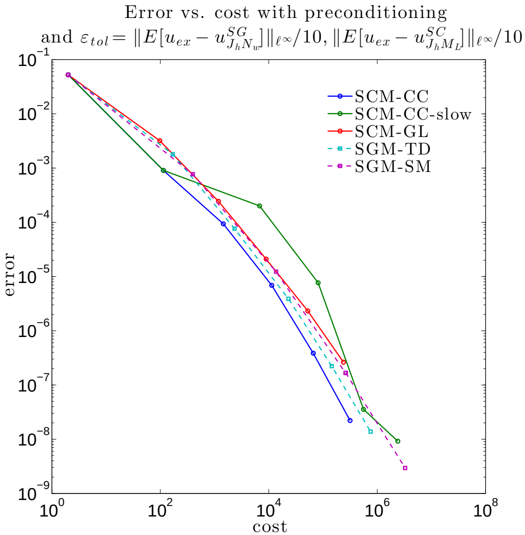

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