A Hierarchical Acceleration of a Stochastic Collocation Method for PDEs with Random Input Data

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Supported by DOE, ASCR via Householder Fellowship and EQUINOX

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July 8, 2014
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Partial differential equations with random input data
A simplified general (stationary) setting

Consider an operator $L$, linear or nonlinear, on a domain $D \subset \mathbb{R}^d$, which depends on some coefficients $a(\omega, x)$ with $x \in D$, $\omega \in \Omega$ and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. The forcing $f = f(\omega, x)$ and the solution $u = u(\omega, x)$ are random fields s.t.

$$L(a)(u) = f \quad \text{a.e. in } D$$

equipped with suitable boundary conditions.

$A_1.$ the solution has realizations in the Banach space $W(D)$, i.e. $u(\cdot, \omega) \in W(D)$ almost surely

$$\|u(\cdot, \omega)\|_{W(D)} \leq C \|f(\cdot, \omega)\|_{W^*(D)}$$

$A_2.$ the forcing term $f \in L^2_{\mathbb{P}}(\Omega; W^*(D))$ is such that the solution $u$ is unique and bounded in $L^2_{\mathbb{P}}(\Omega; W(D))$

$A_3.$ $\mathbb{P}\{a(\omega, x) \in (a_{\min}, a_{\max}), \ \forall x \in \overline{D}\} = 1$ with $0 < a_{\min} < a_{\max} < \infty$
Examples

Linear and nonlinear elliptic PDEs with random input data

Example: The linear elliptic problem

\[
\begin{aligned}
-\nabla \cdot (a(\omega, \cdot) \nabla u(\omega, \cdot)) &= f(\omega, \cdot) \quad \text{in } \Omega \times D, \\
u(\omega, \cdot) &= 0 \quad \text{on } \Omega \times \partial D,
\end{aligned}
\]

with \( f(\omega, \cdot) \) square integrable with respect to \( \mathbb{P} \), satisfies assumptions \( A_1, A_2 \) and \( A_3 \) with \( W(D) = H^1_0(D) \).

Example: The nonlinear elliptic problem

Similarly, for \( k \in \mathbb{N}^+ \),

\[
\begin{aligned}
-\nabla \cdot (a(\omega, \cdot) \nabla u(\omega, \cdot)) + u(\omega, \cdot)|u(\omega, \cdot)|^k &= f(\omega, \cdot) \quad \text{in } \Omega \times D, \\
u(\omega, \cdot) &= 0 \quad \text{on } \Omega \times \partial D,
\end{aligned}
\]

satisfies assumptions \( A_1, A_2 \) and \( A_3 \) with \( W(D) = H^1_0(D) \cap L^{k+2}(D) \).
Parameterization of random fields

Finite dimensional noise assumption

WLOG assume the random fields $a(\omega, x)$ and $f(\omega, x)$ depends on a finite number of random variables $y(\omega) = [y_1(\omega), \ldots, y_N(\omega)] : \Omega \rightarrow \mathbb{R}^N$ s.t.

$$a(\omega, x) = a(y(\omega), x), \quad f(\omega, x) = f(y(\omega), x)$$

- **Piecewise constant approximations**: let $\{D_n\}_{n=1}^N$ be a partition of $D$ then define $a(\omega, x) = \sum_{n=1}^N \sigma_n y_n(\omega) \chi_{D_n}(x)$

- **Truncated infinite dimensional expansions**: e.g. expand $a(\omega, x)$ in a Karhunen-Loève expansion and retain the first $N$ terms.

- Given $a(y(\omega), x), f(y(\omega), x) \Rightarrow u(\omega, x) = u(y_1(\omega), \ldots, y_N(\omega), x)$ s.t.

  $$\mathcal{L}(a(y, x))(u(y, x)) = f(y, x) \quad \text{in } D \ \text{a.s.}$$

- $y(\omega)$ has a joint PDF $\rho : \Gamma \equiv \prod_{n=1}^N \Gamma_n \subset \mathbb{R}^N \rightarrow \mathbb{R}^+,$ with $\rho \in L^\infty(\Gamma)$

- The probability space $(\Omega, \mathcal{F}, P)$ is mapped to $(\Gamma, \mathcal{B}(\Gamma), \rho(y)dy), \text{where } \mathcal{B}(\Gamma) \text{ denotes the Borel } \sigma\text{-algebra on } \Gamma \text{ and } \rho(y)dy \text{ is the probability measure of } y$
Stochastic non-intrusive methods
Multivariate approximations of \( u(y, x) \) in \( \Gamma \times D \)

The general procedure of a non-intrusive approach of parametric PDEs is

- Choose a set of sample points \( \mathcal{H}_M(\Gamma) = \{ y_k \in \Gamma \}_{k=1}^M \)
- Solve the parametric PDE system to obtain \( u_{N_h}(y_k) \) for \( k = 1, \ldots, M \) by using deterministic solvers with mesh size \( h \)
- Construct a multivariate approximation with the sampled values:

\[
 u_{N_h,M}(y, x) = \sum_{k=1}^M c_k(x) \cdot \psi_k(y),
\]

where, for \( k = 1, \ldots, M \), \( u_{N_h,M}(y_k, x) = u_{N_h}(y_k, x) \), \( \psi_k(y) \in \mathcal{P}(\Gamma) \) are basis functions which expand the polynomial space \( \mathcal{P}(\Gamma) \), and the coefficients \( c_k(x) \) are determined by the samples \( \{ u_{N_h}(y_k, x) \}_{k=1}^M \)

Quantity of interest, e.g. \( \mathbb{E}[u](x) \)

\[
 \mathbb{E}[u](x) \approx \int_{\Gamma} u_{N_h,M}(y, x) \rho(y) dy = \sum_{k=1}^M c_k(x) \int_{\Gamma} \psi_k(y) \rho(y) dy
\]
Examples of non-intrusive methods

- Monte Carlo and quasi Monte Carlo methods
  - low regularity requirement of $u(y, x)$
  - slow convergence, e.g. $O(M^{1/2})$

- Non-intrusive polynomial chaos expansions
  - high regularity requirement of $u(y, x)$
  - orthogonal polynomial basis
  - fast convergence rates

- Global (sparse-grid) stochastic collocation methods
  - high regularity requirement of $u(y, x)$
  - fast convergence rates
  - dimensional adaptivity

- Piecewise hierarchical (sparse-grid) stochastic collocation methods
  - low regularity requirement of $u(y, x)$
  - faster convergence than Monte Carlo methods
  - local adaptivity
Hierarchical stochastic collocation methods
Complexity reduction through solution acceleration

At level $L \in \mathbb{N}_+$, a hierarchical SC approximation is (informally) defined by:

$$u_{N_h, M_L} = \mathcal{I}_L[u_{N_h}] = \mathcal{I}_{L-1}[u_{N_h}] + \Delta_L[u_{N_h}]$$

- $\mathcal{I}_{L-1}[u_{N_h}]$ is the hierarchical interpolant at level $L - 1$, and $\Delta_L[u_{N_h}]$ is the corresponding hierarchical incremental interpolant at level $L$.

- $\Delta_L[u_{N_h}]$ interpolates on the difference grid $\Delta \mathcal{H}_{M_L} = \mathcal{H}_{M_L} \setminus \mathcal{H}_{M_{L-1}}$, s.t. $\Delta_L[u_{N_h}] \to 0$ as $L \to \infty$.

Relationship to multilevel methods (e.g., MLMC, MLSC): reduce complexity by balancing errors across a sequence of spatial approximations [TJWG2014]

$$u_{K}^{(ML)} := \sum_{k=0}^{K} \mathcal{I}_{L_{K-k}} \left[ u_{N_{h_k}} - u_{N_{h_{k-1}}} \right]$$
Local hierarchical sparse-grid approximation of $u(y, x)$

Sparse grid construction

The hierarchical sparse-grid interpolant $u_{N_h, M_L}(y, x)$ is defined by

$$u_{N_h, M_L}(y, x) = \sum_{l=0}^{L} \sum_{g(l) = l} \sum_{i \in B_1} c_{l,i}(x) \cdot \psi_{l,i}(y)$$

- $\psi_{l,i}(y) = \prod_{n=1}^{N} \psi_{i_n}(y_n)$
- $c_{l,i}(x)$ is the surplus
- $B_1$ is the index set of the “block” $l = (l_1, \ldots, l_N)$
- $L$ indicates the resolution level of $u_{N_h, M_L}(y, x)$ in $\Gamma$
- Full tensor-product grid:
  $$g(l) = \max_{n=1, \ldots, N} l_n$$
- Isotropic sparse grid:
  $$g(l) = l_1 + \cdots + l_N$$

Construction of the surpluses $c_{1,i}(x)$ in $u_{N_h,M_L}(y,x)$

- The surpluses are given in terms of the spatial finite element basis $\{\phi_j(x)\}_{j=1}^{N_h}$ by $c_{1,i}(x) = \sum_{j=1}^{N_h} c_{j,1,i} \cdot \phi_j(x)$ such that

$$u_{N_h,M_L}(y,x) = \sum_{j=1}^{N_h} \left( \sum_{l=0}^{L} \sum_{|l| = l} \sum_{i \in B_1} c_{j,1,i} \cdot \psi_{1,i}(y) \right) \phi_j(x)$$

- For $c_{1,i}(x)$ with $l = L$, the hierarchical interpolatory property provides us

$$c_{j,1,i} = u_{N_h}(y_{1,i}, x_j) - u_{N_h,M_{L-1}}(y_{1,i}, x_j) \quad \text{for} \quad j = 1, \ldots, N_h$$

- When $u(y,x)$ has bounded mixed second-order weak derivatives with respect to $y$, the surpluses can be bounded as

$$|c_{j,1,i}| \leq C2^{-2|l|} \quad \text{for} \quad i \in B_1 \text{ and } j = 1, \ldots, N_h$$

- The hierarchical structure, especially the decay of surpluses, immediately leads to a strategy to do adaptivity
Two-dimensional adaptive sparse grid
Level 0, 1, 2 sparse grids with \( l_1 + l_2 \leq 2 \)

Each point that corresponds to a large surplus, e.g., the points in red, blue, or green, lead to 2 children points added in each direction resulting in the adaptive sparse grid
Computational cost of stochastic collocation methods
for complex PDE systems

- At each collocation point $y_{1,i}$, $u_{Nh}(x, y_{1,i})$ is approximated based on the solution from the selected linear system solver, i.e.

$$u_{Nh}(x, y_{1,i}) = \sum_{j=1}^{Nh} u_{j,1,i} \phi_j(x) \approx \tilde{u}_{Nh}(x, y_{1,i}) = \sum_{j=1}^{Nh} \tilde{u}_{j,1,i} \phi_j(x)$$

where $\tilde{u}_{1,i} = (\tilde{u}_{1,1,i}, \ldots, \tilde{u}_{Nh,1,i})^T$ is the output of the solver.

- In the case of using conjugate gradient methods, the error $e_{1,i}^k = u_{1,i} - u_{1,i}^k$ is bounded by

$$\|e_{1,i}^k\|_{A_{1,i}} \leq 2 \left( \frac{\sqrt{\kappa_{1,i}} - 1}{\sqrt{\kappa_{1,i}} + 1} \right)^k \|e_{1,i}^0\|_{A_{1,i}}$$

- We describe the total computational cost for constructing $\tilde{u}_{Nh,ML} \approx u_{Nh,ML}$ is represented by

$$C_{total} = \sum_{l=0}^{L} \sum_{|l|=l} \sum_{i \in B_l} M_{1,i}$$

where $M_{1,i}$ is the number of iterations needed at the collocation point $y_{1,i}$. 
The approximation \( \tilde{u}_{N_h,M_L}(x,y) \) can be represented in a hierarchical manner,

\[
\tilde{u}_{N_h,M_L}(x,y) = \tilde{u}_{N_h,M_{L-1}}(x,y) + \sum_{g(1)=L} \sum_{i \in B_l} \tilde{c}_{1,i}(x) \cdot \psi_{1,i}(y)
\]

At each collocation point \( y_{1,i} \) on level \( L \), \( u_{1,i} = (u_{1,1,i}, \ldots, u_{N_h,1,i})^\top \) can be represented by

\[
u_{j,1,i} = u_{N_h,M_{L-1}}(x_j, y_{1,i}) + c_{j,1,i}, \text{ for } j = 1, \ldots, N_h
\]

**Key idea**

Due to the decay of \( |c_{j,1,i}| \) as \(|l| \to \infty\), the initial guess for the CG solver is given by

\[
\tilde{u}^0_{1,i} = (\tilde{u}_{N_h,M_{L-1}}(x_1, y_{1,i}), \ldots, \tilde{u}_{N_h,M_{L-1}}(x_{N_h}, y_{1,i}))^\top
\]

where the error of such prediction is, for \( j = 1, \ldots, N_h \),

\[
|\tilde{u}^0_{j,1,i} - u(x_j, y_{1,i})| \leq |\tilde{u}_{N_h,M_{L-1}}(x_j, y_{1,i}) - u_{N_h,M_{L-1}}(x_j, y_{1,i})| + c_{j,1,i}
\]
What about the “total” error? 
when accounting for FEM, sparse-grid and solver errors

- The total error $e = u(x, y) - \tilde{u}_{N_h, M_L}(x, y)$ can be split into

$$
\|e\| \leq \|u - u_{N_h}\| + \|u_{N_h} - u_{N_h, M_L}\| + \|u_{N_h, M_L} - \tilde{u}_{N_h, M_L}\|
$$

$e_1$ (FEM error) $e_2$ (SG error) $e_3$ (solver error)

**Lemma [GWZ14]: total error estimate**

For the second-order elliptic PDE with homogeneous Dirichlet boundary conditions, the approximate solution $\tilde{u}_{N_h, M_L}$ is constructed using the piecewise linear SG method, and the CG solver. Then the error $e = u - \tilde{u}_{N_h, M_L}$ is bounded by

$$
\|e\|_{L^2(D \times \Gamma)} \leq \underbrace{C_{fem}h^{r+1}}_{\text{bound of } e_1} + \underbrace{C_{sg}2^{-2L} \sum_{k=0}^{N-1} \binom{L + N - 1}{k}}_{\text{bound of } e_2} + \underbrace{2^N \binom{L + N}{N} e_{cg}}_{\text{bound of } e_3}
$$

where $u \in H^{r+1}(D) \otimes L^2_\rho(\Gamma)$, the constant $C_{fem}$ is independent of $h$ and $y$, the constant $C_{sg}$ is independent of $L$ and $N$, and $e_{cg}$ is the maximum CG error.
The goal is to estimate the computational cost of constructing $\tilde{u}_{N_h,M_L}$ with the prescribed accuracy $\|e\| = \|u_{N_h,M_L} - \tilde{u}_{N_h,M_L}\| \leq \varepsilon$

According to the error estimate, a sufficient condition of $\|e\| \leq \varepsilon$ is as follows:

\[\|e_1\| \leq C_{\text{fem}} \cdot h^{r+1} \leq \frac{\varepsilon}{3}\]

\[\|e_2\| \leq C_{\text{sg}} \cdot 2^{-2L} \sum_{k=0}^{N-1} \binom{L + N - 1}{k} \leq \frac{\varepsilon}{3}\]

\[\|e_3\| \leq 2^N \binom{L + N}{N} e_{\text{cg}} \leq \frac{\varepsilon}{3}\]

Let $C_{\text{min}}$ represents the minimum cost, i.e. the minimum number of conjugate gradient iterations, to satisfy the above inequalities, and we are trying to estimate an upper bound of $C_{\text{min}}$.

- Estimate maximum $h$ and minimum $L$ to achieve the bounds for $e_1$ and $e_2$
- Determine the necessary value of $e_{\text{cg}}$ to achieve the bound for $e_3$
- Estimate the upper bound for $C_{\text{min}}$ based on $e_{\text{cg}}$
Computational cost analyses
for $\varepsilon$-complexity

- It is straightforward to obtain the upper bound of $h$, i.e.,

$$h \leq \left( \frac{\varepsilon}{3C_{fem}} \right)^{\frac{1}{r+1}}$$

- The upper bound of the minimum level $L$ to achieve $e_2 \leq \frac{\varepsilon}{3}$ is given by

**Lemma [GWZ14, WW95]**

For $\varepsilon < 3C_{sg}$, the accuracy $\|e_2\| \leq \frac{\varepsilon}{3}$ can be achieved with level $L_{\text{min}}$ bounded by

$$L_{\text{min}} \leq L_k = \frac{t_k N}{2 \ln 2} + 1 \quad \text{with} \quad s = \frac{2e}{\ln 2} \left( \frac{3C_{cg}}{\varepsilon} \right)$$

where $\{t_k\}_{k=0}^{\infty}$ is a monotonically decreasing sequence defined by

$$t_k = \ln(t_{k-1} s) \quad \text{with} \quad t_0 = \frac{e}{e - 1} \ln s.$$
The maximum CG error $e_{cg}$ can be bounded by

$$e_{cg} = \max_{i \in B_1, |l| \leq L} \|e_{1,i}\|_2 \leq 2\sqrt{\kappa} \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^J \cdot \tau_0$$

with

$$\kappa = \max_{i \in B_1, |l| \leq L} \kappa_{1,i} \leq \bar{\kappa}, \quad J = \max_{i \in B_1, |l| \leq L} J_{1,i}, \quad \tau_0 = \max_{i \in B_1, |l| \leq L} \|e_{0,i}\|$$

$\kappa$ is the maximum condition number of all the involved linear systems which can be bounded by a function of $1/h$ or $1/\varepsilon$.

The minimum iteration number $J$ to achieve $e_3 \leq \frac{\varepsilon}{3}$ for specific $\tau_0$, $\varepsilon$, $\bar{\kappa}$, $L$ and $N$ is bounded by

$$J(\tau_0, \varepsilon, \bar{\kappa}, L, N) \leq \frac{1}{2} \log_2(\bar{\kappa}) + \log_2 \left[ \frac{3.2^{N+1} \tau_0}{\varepsilon} (\frac{L+N}{N}) \right] \log_2 \left( \frac{\sqrt{\bar{\kappa}+1}}{\sqrt{\bar{\kappa}}-1} \right)$$
Computational cost analyses without hierarchical acceleration

- Without hierarchical acceleration, $\tau_0 = \mathcal{O}(\|u\|_\infty)$, so that the minimum cost $C_{\text{min}}$ to achieve $\|e\| \leq \varepsilon$ can be bounded by

$$C_{\text{min}} \leq |\mathcal{H}_L(\Gamma)| \cdot J(\tau_0, \varepsilon, \kappa, L_k, N)$$

whose estimate is given as follows:

**Theorem [GWZ14], complexity without hierarchical acceleration**

The minimum cost $C_{\text{min}}$ for building the standard piecewise linear SG approximation $\tilde{u}_{N_h,M_L}(x,y)$ with the prescribed accuracy $\varepsilon > 0$ can be bounded by

$$C_{\text{min}} \leq \frac{\alpha_1}{N} \left[ \alpha_2 + \alpha_3 \log_2 \left( \frac{3C_{\text{sg}}}{\varepsilon} \right) \right]^{\alpha_4 N} \left( \frac{3C_{\text{sg}}}{\varepsilon} \right)^{\alpha_5} \times \frac{1}{\log_2 \left( \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right)} \left[ \alpha_6 \log_2 \left( \frac{3C_{\text{sg}}}{\varepsilon} \right) + \log_2(\sqrt{\kappa}\tau_0) + \alpha_7 N + \alpha_8 \right],$$

where the constants $\alpha_1, \ldots, \alpha_8$ are independent of $L$, $N$ and $\varepsilon$. 
With hierarchical acceleration, $\tau_0^l \leq C_{sg}2^{-2l} + 2^Ne_{cg}$ for $l = 1, \ldots, L$, so that the minimum cost $C_{\text{min}}$ to achieve $\|e\| \leq \varepsilon$ can be bounded by

$$C_{\text{min}} \leq \sum_{l=0}^{L_k} |\Delta \mathcal{H}_l(\Gamma)| \cdot J(\tau_0^l, \varepsilon, \kappa, L_k, N)$$

whose estimate is given as follows:

Theorem [GWZ14], complexity with hierarchical acceleration

The minimum cost $C_{\text{min}}$ for building the standard piecewise linear SG approximation $\tilde{u}_{N_h,M_L}(x,y)$ with the prescribed accuracy $\varepsilon > 0$ can be bounded by

$$C_{\text{min}} \leq \alpha_1 \left[ \alpha_2 + \alpha_3 \frac{\log_2 \left( \frac{2C_{sg}}{\varepsilon} \right)}{N} \right]^{\alpha_4 N} \left( \frac{2C_{sg}}{\varepsilon} \right)^{\alpha_5}$$

$$\times \frac{1}{\log_2 \left( \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right)} \left[ 2N - \log_2 (N) + \alpha_9 + \log_2 (\sqrt{\kappa}) \right],$$

where the constants $\alpha_1, \ldots, \alpha_5$ and $\alpha_9$ are independent of $L$, $N$ and $\varepsilon$. 
Numerical example
Case 1: linear elliptic problem with random inputs

We consider the 2D Poisson equation with random diffusivity and forcing term, i.e.,

\[
\begin{cases}
\nabla \cdot (a(x, y) \nabla u(x, y)) = f(x, y) & \text{in } [0, 1]^2 \times \Gamma, \\
u(x, y) = 0 & \text{on } \partial D \times \Gamma,
\end{cases}
\]

where \(a\) and \(f\) are the nonlinear functions of the random vector \(y\) given by

\[a(x, y) = 0.1 + \exp \left[ y_1 \cos(\pi x_1) + y_2 \sin(\pi x_2) \right],\]

and

\[f(x, y) = 10 + \exp \left[ y_3 \cos(\pi x_1) + y_4 \sin(\pi x_2) \right],\]

where \(y_n\) for \(n = 1, 2, 3, 4\) are i. i. d. random variables following the uniform distribution \(U([-1, 1])\). The quantity of interest is the mean value of the solution over \(D \times \Gamma\), i.e.,

\[QoI = \mathbb{E} \left[ \int_D u(x, y) dx \right],\]
### Numerical example

**Case 1: linear elliptic problem with random inputs**

The computational savings of the piecewise SG approach with hierarchical acceleration

<table>
<thead>
<tr>
<th>Basis type</th>
<th>Error</th>
<th># SG points</th>
<th>hSGSC cost</th>
<th>hSGSC+acceleration cost</th>
<th>saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>1.0e-2</td>
<td>377</td>
<td>13,841</td>
<td>7,497</td>
<td>45.8%</td>
</tr>
<tr>
<td></td>
<td>1.0e-3</td>
<td>1,893</td>
<td>81,068</td>
<td>38,670</td>
<td>52.2%</td>
</tr>
<tr>
<td></td>
<td>1.0e-4</td>
<td>7,777</td>
<td>376,287</td>
<td>167,832</td>
<td>55.3%</td>
</tr>
<tr>
<td>Quadratic</td>
<td>1.0e-3</td>
<td>701</td>
<td>29,874</td>
<td>11,877</td>
<td>60.2%</td>
</tr>
<tr>
<td></td>
<td>1.0e-4</td>
<td>2,285</td>
<td>110,744</td>
<td>36,760</td>
<td>66.8%</td>
</tr>
<tr>
<td></td>
<td>1.0e-5</td>
<td>6,149</td>
<td>329,294</td>
<td>100,420</td>
<td>69.5%</td>
</tr>
<tr>
<td>Cubic</td>
<td>1.0e-4</td>
<td>1,233</td>
<td>59,344</td>
<td>23,228</td>
<td>60.8%</td>
</tr>
<tr>
<td></td>
<td>1.0e-5</td>
<td>3,233</td>
<td>172,845</td>
<td>57,777</td>
<td>66.5%</td>
</tr>
<tr>
<td></td>
<td>1.0e-6</td>
<td>7,079</td>
<td>415,760</td>
<td>129,433</td>
<td>68.8%</td>
</tr>
</tbody>
</table>
Let $\mathbf{x} = (x_1, x_2)$ and consider the following linear elliptic SPDE:

$$\begin{align*}
-\nabla \cdot (a(x_1, y(\omega))\nabla u(x, y)) &= \cos(x_1) \sin(x_2) & [0, 1]^2 \times \Gamma \\
u(x, y) &= 0 & \text{on } \partial D \times \Gamma
\end{align*}$$

The diffusion coefficient is a 1d random field (varies only in $x_1$) and is

$$a(x_1, y) = 0.5 + \exp\{\gamma(x_1, y)\},$$

where $\gamma$ is a truncated 1d random field with correlation length $L$ and covariance

$$Cov[\gamma](x_1, \tilde{x}_1) = \exp\left(-\frac{(x_1 - \tilde{x}_1)^2}{L^2}\right), \quad \forall (x_1, \tilde{x}_1) \in [0, 1]$$

$$\gamma(x_1, y) = 1 + y_1(\omega) \left(\frac{\sqrt{\pi L}}{2}\right)^{1/2} + \sum_{n=2}^{N} \beta_n \varphi_n(x_1) y_n(\omega)$$

$$\beta_n := \left(\frac{\sqrt{\pi L}}{2}\right)^{1/2} e^{-\frac{\left\lfloor \frac{n}{2} \right\rfloor \pi L}{8}}, \quad \varphi_n(x_1) := \begin{cases} 
\sin \left(\frac{n}{2} \pi x_1\right), & \text{if } n \text{ even}, \\
\cos \left(\frac{n}{2} \pi x_1\right), & \text{if } n \text{ odd}
\end{cases}$$

- $\mathbb{E}[y_n] = 0$ and $\mathbb{E}[y_n y_m] = \delta_{nm}$ for $n, m \in \mathbb{N}_+$ and iid in $U(-\sqrt{3}, \sqrt{3})$
Computational savings

\[ \text{QoI} = \mathbb{E}[u] \]

Savings versus level and savings versus error for \( L = 1/64 \) (left) and \( L = 1/2 \) (right)
Conclusion and future work

The bulk of the computational cost of high-dimensional extreme-scale computational simulations is associated with linear or nonlinear iterative solvers, and the convergence of such methods can be dramatically improved by using the hierarchical acceleration approach.

The new method can be easily extended to other non-intrusive methods, including global sparse-grid approximation, polynomial chaos expansion, etc..

Extend the new approach to time-dependent problems

Incorporate this approach into a multi-level framework using model hierarchies \( \{u^s(x, y)\}_{s=1}^S \) of increasing complexity.

\[
\|\hat{u}^S - u^S\| \leq \|u^0_{M_L} - u^0\| + \sum_{s=1}^S \|(u^s_{M_L - s} - u^{s-1}_{M_L - s}) - (u^s - u^{s-1})\| \leq \epsilon
\]
References


