A Hierarchical Acceleration of a Stochastic Collocation Method for PDEs with Random Input Data

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Outline

1. Stochastic problems: PDEs with random inputs
2. Stochastic non-intrusive methods (MC, NIPC, SC)
3. Piecewise adaptive sparse-grid approximation with hierarchical acceleration
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Partial differential equations with random input data

A simplified general (stationary) setting

Consider an operator $\mathcal{L}$, linear or nonlinear, on a domain $D \subset \mathbb{R}^d$, which depends on some coefficients $a(\omega, x)$ with $x \in D$, $\omega \in \Omega$ and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. The forcing $f = f(\omega, x)$ and the solution $u = u(\omega, x)$ are random fields s.t.

$$\mathcal{L}(a)(u) = f \quad \text{a.e. in } D$$

equipped with suitable boundary conditions.

A_1. the solution has realizations in the Banach space $W(D)$, i.e. $u(\cdot, \omega) \in W(D)$ almost surely

$$\|u(\cdot, \omega)\|_{W(D)} \leq C \|f(\cdot, \omega)\|_{W^*(D)}$$

A_2. the forcing term $f \in L^2_{\mathbb{P}}(\Omega; W^*(D))$ is such that the solution u is unique and bounded in $L^2_{\mathbb{P}}(\Omega; W(D))$

A_3. $\mathbb{P}\left\{a(\omega, x) \in (a_{\min}, a_{\max}), \ \ \forall x \in \overline{D}\right\} = 1$ with $0 < a_{\min} < a_{\max} < \infty$

Examples

Linear and nonlinear elliptic PDEs with random input data

Example: The linear elliptic problem

$$\begin{cases} -\nabla \cdot (a(\omega,\cdot)\nabla u(\omega,\cdot)) &= f(\omega,\cdot) \quad \text{in } \Omega \times D, \\ u(\omega,\cdot) &= 0 \quad \text{on } \Omega \times \partial D, \end{cases}$$

with $f(\omega,\cdot)$ square integrable with respect to $\mathbb{P}$, satisfies assumptions A_1, A_2 and A_3 with $W(D) = H_0^1(D)$

Example: The nonlinear elliptic problem

Similarly, for $k \in \mathbb{N}^+$,

$$\begin{cases} -\nabla \cdot (a(\omega,\cdot)\nabla u(\omega,\cdot)) + u(\omega,\cdot)|u(\omega,\cdot)|^k &= f(\omega,\cdot) \quad \text{in } \Omega \times D, \\ u(\omega,\cdot) &= 0 \quad \text{on } \Omega \times \partial D, \end{cases}$$

satisfies assumptions A_1, A_2 and A_3 with $W(D) = H_0^1(D) \cap L^{k+2}(D)$

Parameterization of random fields

Finite dimensional noise assumption

WLOG assume the random fields $a(\omega, x)$ and $f(\omega, x)$ depends on a finite number of random variables $\boldsymbol{y}(\omega) = [y_1(\omega), \ldots, y_N(\omega)] : \Omega \to \mathbb{R}^N$ s.t.

$$a(\omega, x) = a(\boldsymbol{y}(\omega), x), \quad f(\omega, x) = f(\boldsymbol{y}(\omega), x)$$

- Piecewise constant approximations: let $\{D_n\}_{n=1}^N$ be a partition of D then define $a(\omega, x) = \sum_{n=1}^N \sigma_n y_n(\omega) \chi_{D_n}(x)$
- Truncated infinite dimensional expansions: e.g. expand $a(\omega, x)$ in a Karhunen-Loève expansion and retain the first N terms.

- Given $a(\boldsymbol{y}(\omega), x)$, $f(\boldsymbol{y}(\omega), x) \Rightarrow u(\omega, x) = u(y_1(\omega), \ldots, y_N(\omega), x)$ s.t.

$$\mathcal{L}(a(\boldsymbol{y}, x))(u(\boldsymbol{y}, x)) = f(\boldsymbol{y}, x) \quad \text{in } D \text{ a.s.}$$

- $\boldsymbol{y}(\omega)$ has a joint PDF $\rho : \Gamma \equiv \prod_{n=1}^N \Gamma_n \subset \mathbb{R}^N \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma)$
- The probability space $(\Omega, \mathcal{F}, P)$ is mapped to $(\Gamma, \mathcal{B}(\Gamma), \rho(\boldsymbol{y})d\boldsymbol{y})$, where $\mathcal{B}(\Gamma)$ denotes the Borel σ-algebra on Γ and $\rho(\boldsymbol{y})d\boldsymbol{y}$ is the probability measure of $\boldsymbol{y}$

Stochastic non-intrusive methods

Multivariate approximations of $u(\boldsymbol{y}, x)$ in $\Gamma \times D$

The general procedure of a non-intrusive approach of parametric PDEs is

- Choose a set of sample points $\mathcal{H}_M(\Gamma) = \{\boldsymbol{y}_k \in \Gamma\}_{k=1}^M$
- Solve the parametric PDE system to obtain $u_{N_h}(\boldsymbol{y}_k)$ for $k = 1, \ldots, M$ by using deterministic solvers with mesh size h
- Construct a multivariate approximation with the sampled values:

$$u_{N_h,M}(\boldsymbol{y}, x) = \sum_{k=1}^{M} c_k(x) \cdot \psi_k(\boldsymbol{y}),$$

where, for $k = 1, \ldots, M$, $u_{N_h,M}(\boldsymbol{y}_k, x) = u_{N_h}(\boldsymbol{y}_k, x)$, $\psi_k(\boldsymbol{y}) \in \mathcal{P}(\Gamma)$ are basis functions which expand the polynomial space $\mathcal{P}(\Gamma)$, and the coefficients $c_k(x)$ are determined by the samples $\{u_{N_h}(\boldsymbol{y}_k, x)\}_{k=1}^M$

Quantity of interest, e.g. $\mathbb{E}[u](x)$

$$\mathbb{E}[u](x) \approx \int_\Gamma u_{N_h,M}(\boldsymbol{y}, x)\rho(\boldsymbol{y})d\boldsymbol{y} = \sum_{k=1}^{M} c_k(x) \int_\Gamma \psi_k(y)\rho(\boldsymbol{y})dy$$

Examples of non-intrusive methods

- Monte Carlo and quasi Monte Carlo methods
  - low regularity requirement of $u(\boldsymbol{y}, x)$
  - slow convergence, e.g. $\mathcal{O}(M^{1/2})$
- Non-intrusive polynomial chaos expansions
  - high regularity requirement of $u(\boldsymbol{y}, x)$
  - orthogonal polynomial basis
  - fast convergence rates
- Global (sparse-grid) stochastic collocation methods
  - high regularity requirement of $u(\boldsymbol{y}, x)$
  - fast convergence rates
  - dimensional adaptivity
- Piecewise hierarchical (sparse-grid) stochastic collocation methods
  - low regularity requirement of $u(\boldsymbol{y}, x)$
  - faster convergence than Monte Carlo methods
  - local adaptivity

Hierarchical stochastic collocation methods

Complexity reduction through solution acceleration

At level $L \in \mathbb{N}_+$, a hierarchical SC approximation is (informally) defined by:

$$u_{N_h,M_L} = \mathcal{I}_L[u_{N_h}] \equiv \mathcal{I}_{L-1}[u_{N_h}] + \Delta_L[u_{N_h}]$$

- $\mathcal{I}_{L-1}[u_{N_h}]$ is the hierarchical interpolant at level $L-1$, and $\Delta_L[u_{N_h}]$ is the corresponding hierarchical incremental interpolant at level L
- $\Delta_L[u_{N_h}]$ interpolates on the difference grid $\Delta\mathcal{H}_{M_L} = \mathcal{H}_{M_L} \backslash \mathcal{H}_{M_{L-1}}$, s.t. $\Delta_L[u_{N_h}] \to 0$ as $L \to \infty$

Relationship to multilevel methods (e.g., MLMC, MLSC): reduce complexity by balancing errors across a sequence of spatial approximations [TJWG2014]

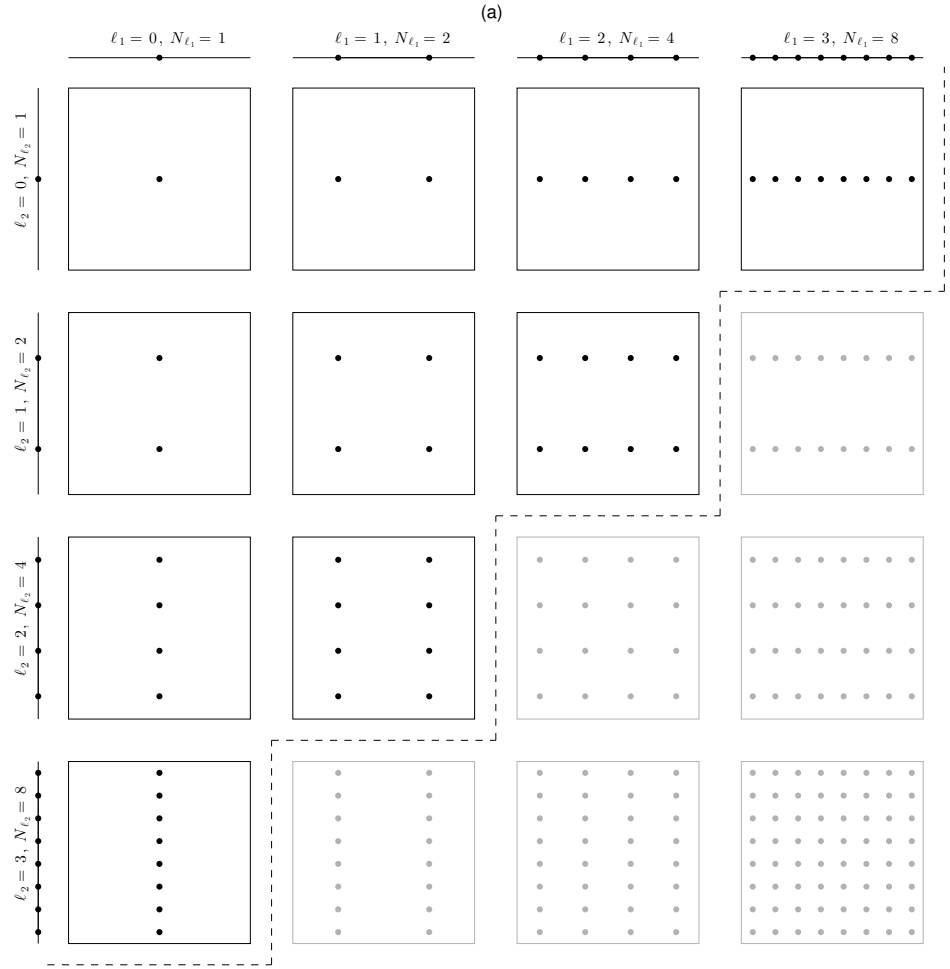
$$u_K^{(\mathrm{ML})} := \sum_{k=0}^{K} \mathcal{I}_{L_{K-k}} \left[u_{N_{h_k}} - u_{N_{h_{k-1}}} \right]$$

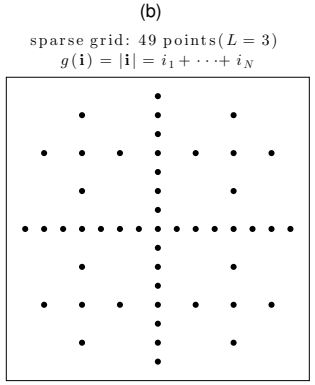
Local hierarchical sparse-grid approximation of $u(\boldsymbol{y}, x)$

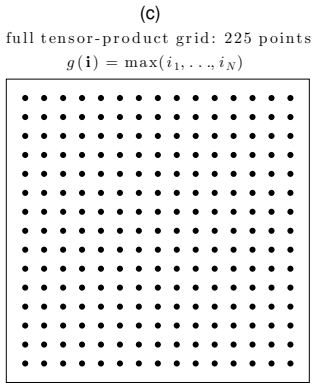
Sparse grid construction

The hierarchical sparse-grid interpolant $u_{N_h,M_L}(\boldsymbol{y}, x)$ is defined by

$$u_{N_h,M_L}(\boldsymbol{y}, x) = \sum_{l=0}^{L} \sum_{g(\mathbf{l})=l} \sum_{\mathbf{i}\in B_{\mathbf{l}}} c_{\mathbf{l},\mathbf{i}}(x) \cdot \psi_{\mathbf{l},\mathbf{i}}(\boldsymbol{y})$$







- $\psi_{\mathbf{l},\mathbf{i}}(\boldsymbol{y}) = \prod_{n=1}^{N} \psi_{l_n,i_n}(y_n)$
- $c_{\mathbf{l},\mathbf{i}}(x)$ is the surplus
- $B_{\mathbf{l}}$ is the index set of the "block" $\mathbf{l} = (l_1, \ldots, l_N)$
- L indicates the resolution level of $u_{N_h,M_L}(\boldsymbol{y}, x)$ in Γ
- Full tensor-product grid: $g(\mathbf{l}) = \max_{n=1,\ldots,N} l_n$
- Isotropic sparse grid: $g(\mathbf{l}) = l_1 + \cdots + l_N$

Construction of the surpluses $c_{\mathbf{l},\mathbf{i}}(x)$ in $u_{N_h,M_L}(\boldsymbol{y},x)$

- The surpluses are given in terms of the spatial finite element basis $\{\phi_j(x)\}_{j=1}^{N_h}$ by $c_{\mathbf{l},\mathbf{i}}(x) = \sum_{j=1}^{N_h} c_{j,\mathbf{l},\mathbf{i}} \cdot \phi_j(x)$ such that

$$u_{N_h,M_L}(\boldsymbol{y},x) = \sum_{j=1}^{N_h} \left(\sum_{l=0}^{L} \sum_{|\mathbf{l}|=l} \sum_{\mathbf{i}\in B_{\mathbf{l}}} c_{j,\mathbf{l},\mathbf{i}} \cdot \psi_{\mathbf{l},\mathbf{i}}(\boldsymbol{y}) \right) \phi_j(x)$$

- For $c_{\mathbf{l},\mathbf{i}}(x)$ with $\mathbf{l} = L$, the hierarchical interpolatory property provides us

$$c_{j,\mathbf{l},\mathbf{i}} = u_{N_h}(\boldsymbol{y}_{\mathbf{l},\mathbf{i}}, x_j) - u_{N_h,M_{L-1}}(\boldsymbol{y}_{\mathbf{l},\mathbf{i}}, x_j) \quad \text{for} \quad j = 1,\ldots,N_h$$

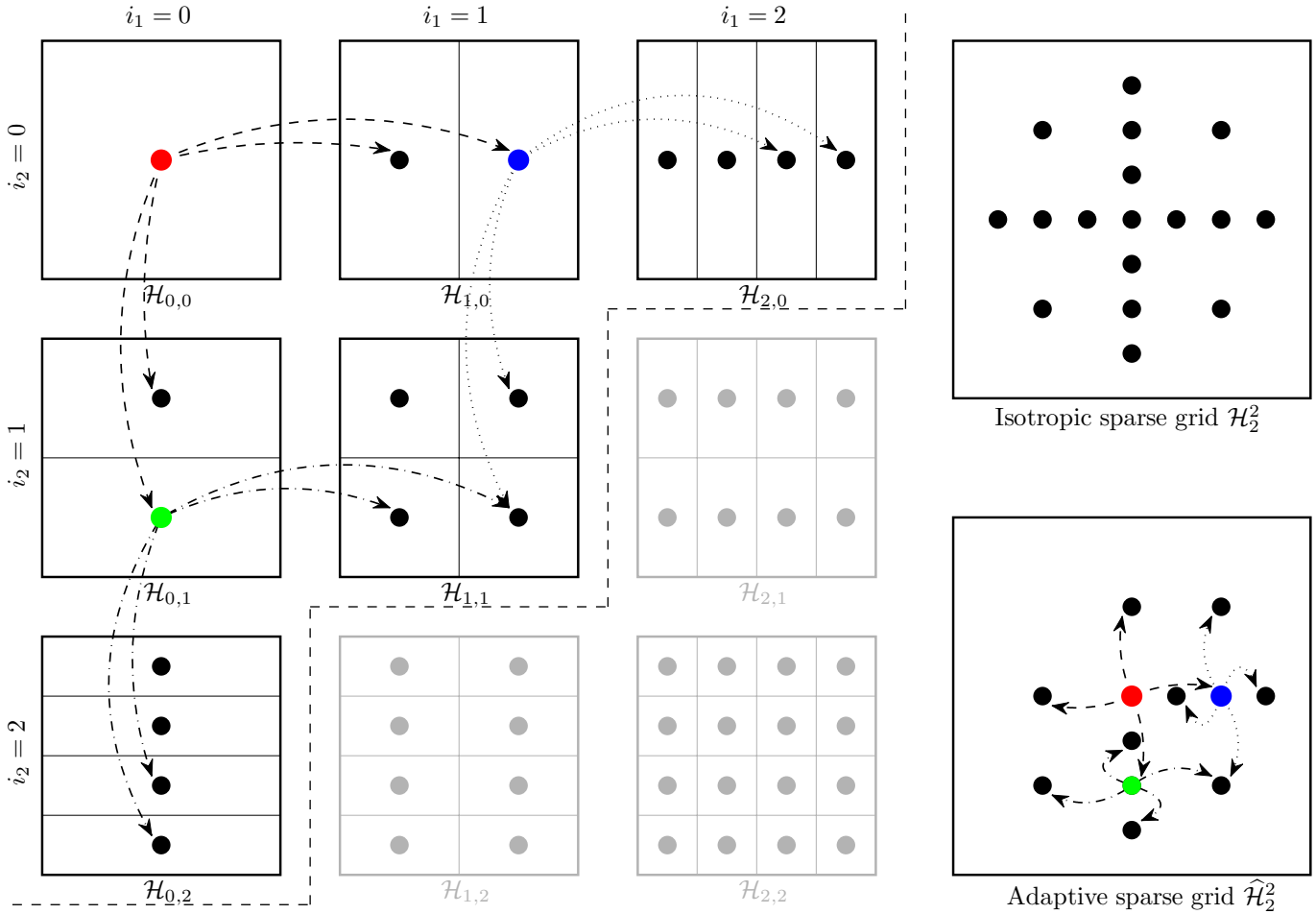
- When $u(\boldsymbol{y},x)$ has bounded mixed second-order weak derivatives with respect to $\boldsymbol{y}$, the surpluses can be bounded as

$$|c_{j,\mathbf{l},\mathbf{i}}| \le C 2^{-2\cdot|\mathbf{l}|} \quad \text{for} \quad \mathbf{i} \in B_{\mathbf{l}} \text{ and } j = 1,\ldots,N_h$$

- The hierarchical structure, especially the decay of surpluses, immediately leads to a strategy to do adaptivity

Two-dimensional adaptive sparse grid

Level $0, 1, 2$ sparse grids with $l_1 + l_2 \leq 2$



Each point that corresponds to a large surplus, e.g., the points in red, blue, or green, lead to 2 children points added in each direction resulting in the adaptive sparse grid

Computational cost of stochastic collocation methods

for complex PDE systems

- At each collocation point $\boldsymbol{y}_{\mathbf{l},\mathbf{i}}$, $u_{N_h}(x, \boldsymbol{y}_{\mathbf{l},\mathbf{i}})$ is approximated based on the solution from the selected linear system solver, i.e.

$$u_{N_h}(x, \boldsymbol{y}_{\mathbf{l},\mathbf{i}}) = \sum_{j=1}^{N_h} u_{j,\mathbf{l},\mathbf{i}} \phi_j(x) \approx \widetilde{u}_{N_h}(x, \boldsymbol{y}_{\mathbf{l},\mathbf{i}}) = \sum_{j=1}^{N_h} \widetilde{u}_{j,\mathbf{l},\mathbf{i}} \phi_j(x)$$

  where $\widetilde{\boldsymbol{u}}_{\mathbf{l},\mathbf{i}} = (\widetilde{u}_{1,\mathbf{l},\mathbf{i}}, \ldots, \widetilde{u}_{N_h,\mathbf{l},\mathbf{i}})^\top$ is the output of the solver.

- In the case of using conjugate gradient methods, the error $\boldsymbol{e}^k_{\mathbf{l},\mathbf{i}} = \boldsymbol{u}_{\mathbf{l},\mathbf{i}} - \boldsymbol{u}^k_{\mathbf{l},\mathbf{i}}$ is bounded by

$$\|\boldsymbol{e}^k_{\mathbf{l},\mathbf{i}}\|_{\boldsymbol{A}_{\mathbf{l},\mathbf{i}}} \leq 2 \left(\frac{\sqrt{\kappa_{\mathbf{l},\mathbf{i}}} - 1}{\sqrt{\kappa_{\mathbf{l},\mathbf{i}}} + 1} \right)^k \|\boldsymbol{e}^0_{\mathbf{l},\mathbf{i}}\|_{\boldsymbol{A}_{\mathbf{l},\mathbf{i}}}$$

- We describe the total computational cost for constructing $\widetilde{u}_{N_h,M_L} \approx u_{N_h,M_L}$ is represented by

$$\mathcal{C}_{\text{total}} = \sum_{l=0}^{L} \sum_{|\mathbf{l}|=l} \sum_{\mathbf{i} \in B_{\mathbf{l}}} \mathcal{M}_{\mathbf{l},\mathbf{i}}$$

  where $\mathcal{M}_{\mathbf{l},\mathbf{i}}$ is the number of iterations needed at the collocation point $\boldsymbol{y}_{\mathbf{l},\mathbf{i}}$.

Exploit the hierarchical structure to accelerate solutions

in the context of local hierarchical sparse-grid approximation

- The approximation $\widetilde{u}_{N_h,M_L}(x,\boldsymbol{y})$ can be represented in a hierarchical manner,

$$\widetilde{u}_{N_h,M_L}(x,\boldsymbol{y}) = \widetilde{u}_{N_h,M_{L-1}}(x,\boldsymbol{y}) + \sum_{g(\mathbf{l})=L} \sum_{\mathbf{i}\in B_{\mathbf{l}}} \widetilde{c}_{\mathbf{l},\mathbf{i}}(x)\cdot\psi_{\mathbf{l},\mathbf{i}}(\boldsymbol{y})$$

- At each collocation point $\boldsymbol{y}_{\mathbf{l},\mathbf{i}}$ on level L, $\boldsymbol{u}_{\mathbf{l},\mathbf{i}} = (u_{1,\mathbf{l},\mathbf{i}},\ldots,u_{N_j,\mathbf{l},\mathbf{i}})^\top$ can be represented by

$$u_{j,\mathbf{l},\mathbf{i}} = u_{N_h,M_{L-1}}(x_j,\boldsymbol{y}_{\mathbf{l},\mathbf{i}}) + c_{j,\mathbf{l},\mathbf{i}}, \quad \text{for} \quad j=1,\ldots,N_h$$

Key idea

Due to the decay of $|c_{j,\mathbf{l},\mathbf{i}}|$ as $|\mathbf{l}|\to\infty$, the initial guess for the CG solver is given by

$$\widetilde{\boldsymbol{u}}^0_{\mathbf{l},\mathbf{i}} = \left(\widetilde{u}_{N_h,M_{L-1}}(x_1,\boldsymbol{y}_{\mathbf{l},\mathbf{i}}),\ldots,\widetilde{u}_{N_h,M_{L-1}}(x_{N_h},\boldsymbol{y}_{\mathbf{l},\mathbf{i}})\right)^\top$$

where the error of such prediction is, for $j=1,\ldots,N_h$,

$$\left|\widetilde{u}^0_{j,\mathbf{l},\mathbf{i}} - u(x_j,\boldsymbol{y}_{\mathbf{l},\mathbf{i}})\right| \le \left|\widetilde{u}_{N_h,M_{L-1}}(x_j,\boldsymbol{y}_{\mathbf{l},\mathbf{i}}) - u_{N_h,M_{L-1}}(x_j,\boldsymbol{y}_{\mathbf{l},\mathbf{i}})\right| + c_{j,\mathbf{l}\,\mathbf{i}}$$

What about the "total" error?

when accounting for FEM, sparse-grid and solver errors

- The total error $e = u(x, \boldsymbol{y}) - \widetilde{u}_{N_h, M_L}(x, \boldsymbol{y})$ can be split into

$$\|e\| \leq \underbrace{\|u - u_{N_h}\|}_{e_1\,(\text{FEM error})} + \underbrace{\|u_{N_h} - u_{N_h, M_L}\|}_{e_2\,(\text{SG error})} + \underbrace{\|u_{N_h, M_L} - \widetilde{u}_{N_h, M_L}\|}_{e_3\,(\text{solver error})}$$

Lemma [GWZ14]: total error estimate

For the second-order elliptic PDE with homogeneous Dirichlet boundary conditions, the approximate solution $\widetilde{u}_{N_h, M_L}$ is constructed using the piecewise linear SG method, and the CG solver. Then the error $e = u - \widetilde{u}_{N_h, M_L}$ is bounded by

$$\|e\|_{L^2(D\times\Gamma)} \leq \underbrace{C_{\text{fem}} h^{r+1}}_{\text{bound of } e_1} + \underbrace{C_{\text{sg}} 2^{-2L} \sum_{k=0}^{N-1} \binom{L+N-1}{k}}_{\text{bound of } e_2} + \underbrace{2^N \binom{L+N}{N} e_{\text{cg}}}_{\text{bound of } e_3}$$

where $u \in H^{r+1}(D) \otimes L^2_\rho(\Gamma)$, the constant C_{fem} is independent of h and $\boldsymbol{y}$, the constant C_{sg} is independent of L and N, and e_{cg} is the maximum CG error.

Computational cost analyses

for ε-complexity

- The goal is to estimate the computational cost of constructing $\widetilde{u}_{N_h,M_L}$ with the prescribed accuracy $\|e\| = \|u_{N_h,M_L} - \widetilde{u}_{N_h,M_L}\| \leq \varepsilon$
- According to the error estimate, a sufficient condition of $\|e\| \leq \varepsilon$ is as follows:

$$\|e_1\| \leq C_{\text{fem}} \cdot h^{r+1} \leq \frac{\varepsilon}{3}$$

$$\|e_2\| \leq C_{\text{sg}} \cdot 2^{-2L} \sum_{k=0}^{N-1} \binom{L+N-1}{k} \leq \frac{\varepsilon}{3}$$

$$\|e_3\| \leq 2^N \binom{L+N}{N} e_{\text{cg}} \leq \frac{\varepsilon}{3}$$

- Let $\mathcal{C}_{\text{min}}$ represents the minimum cost, i.e. the minimum number of conjugate gradient iterations, to satisfy the above inequalities, and we are trying to estimate an upper bound of $\mathcal{C}_{\text{min}}$.
  - Estimate maximum h and minimum L to achieve the bounds for e_1 and e_2
  - Determine the necessary value of e_{cg} to achieve the bound for e_3
  - Estimate the upper bound for C_{min} based on e_{cg}

Computational cost analyses

for ε-complexity

- It is straightforward to obtain the upper bound of h, i.e.,

$$h \leq \left(\frac{\varepsilon}{3C_{\text{fem}}} \right)^{\frac{1}{r+1}}$$

- The upper bound of the minimum level L to achieve $e_2 \leq \frac{\varepsilon}{3}$ is given by

Lemma [GWZ14, WW95]

For $\varepsilon < 3C_{\text{sg}}$, the accuracy $\|e_2\| \leq \frac{\varepsilon}{3}$ can be achieved with level $L_{\min}$ bounded by

$$L_{\min} \leq L_k = \frac{t_k N}{2 \ln 2} + 1 \quad \text{with} \quad s = \frac{2e}{\ln 2} \left(\frac{3C_{\text{cg}}}{\varepsilon} \right)$$

where $\{t_k\}_{k=0}^{\infty}$ is a monotonically decreasing sequence defined by

$$t_k = \ln(t_{k-1} s) \quad \text{with} \quad t_0 = \frac{e}{e-1} \ln s.$$

- $\{L_k\}_{k=0}^{\infty}$ is also a monotonically decreasing sequence

Computational cost analyses

for ε-complexity

- The maximum CG error e_{cg} can be bounded by

$$e_{\text{cg}} = \max_{\mathbf{i}\in B_{\mathbf{l}},|\mathbf{l}|\leq L} \|\boldsymbol{e}_{\mathbf{l},\mathbf{i}}\|_2 \leq 2\sqrt{\kappa}\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^J \cdot \tau_0$$

with

$$\kappa = \max_{\mathbf{i}\in B_{\mathbf{l}},|\mathbf{l}|\leq L} \kappa_{\mathbf{l},\mathbf{i}} \leq \overline{\kappa}, \quad J = \max_{\mathbf{i}\in B_{\mathbf{l}},|\mathbf{l}|\leq L} J_{\mathbf{l},\mathbf{i}}, \quad \tau_0 = \max_{\mathbf{i}\in B_{\mathbf{l}},|\mathbf{l}|\leq L} \|e^0_{\mathbf{l},\mathbf{i}}\|$$

- κ is the maximum condition number of all the involved linear systems which can be bounded by a function of $1/h$ or $1/\varepsilon$.

- The minimum iteration number J to achieve $e_3 \leq \frac{\varepsilon}{3}$ for specific τ_0, ε, $\overline{\kappa}$, L and N is bounded by

$$J(\tau_0,\varepsilon,\overline{\kappa},L,N) \leq \frac{\frac{1}{2}\log_2(\overline{\kappa}) + \log_2\left[\frac{3\cdot 2^{N+1}\tau_0}{\varepsilon}\binom{L+N}{N}\right]}{\log_2\left(\frac{\sqrt{\overline{\kappa}}+1}{\sqrt{\overline{\kappa}}-1}\right)}$$

Computational cost analyses

without hierarchical acceleration

- Without hierarchical acceleration, $\tau_0 = \mathcal{O}(\|u\|_\infty)$, so that the minimum cost $\mathcal{C}_{\min}$ to achieve $\|e\| \leq \varepsilon$ can be bounded by

$$\mathcal{C}_{\min} \leq |\mathcal{H}_L(\Gamma)| \cdot J(\tau_0, \varepsilon, \overline{\kappa}, L_k, N)$$

whose estimate is given as follows:

Theorem [GWZ14], complexity without hierarchical acceleration

The minimum cost $\mathcal{C}_{\min}$ for building the standard piecewise linear SG approximation $\widetilde{u}_{N_h,M_L}(x,\boldsymbol{y})$ with the prescribed accuracy $\varepsilon > 0$ can be bounded by

$$\mathcal{C}_{\min} \leq \frac{\alpha_1}{N}\left[\alpha_2 + \alpha_3 \frac{\log_2\left(\frac{3C_{\mathrm{sg}}}{\varepsilon}\right)}{N}\right]^{\alpha_4 N} \left(\frac{3C_{\mathrm{sg}}}{\varepsilon}\right)^{\alpha_5}$$
$$\times \frac{1}{\log_2\left(\frac{\sqrt{\overline{\kappa}}+1}{\sqrt{\overline{\kappa}}-1}\right)}\left[\alpha_6 \log_2\left(\frac{3C_{\mathrm{sg}}}{\varepsilon}\right) + \log_2(\sqrt{\overline{\kappa}}\tau_0) + \alpha_7 N + \alpha_8\right],$$

where the constants $\alpha_1, \ldots, \alpha_8$ are independent of L, N and ε.

Computational cost analyses

with hierarchical acceleration

- With hierarchical acceleration, $\tau_0^l \leq C_{\text{sg}} 2^{-2l} + 2^N e_{\text{cg}}$ for $l = 1, \ldots, L$, so that the minimum cost $\mathcal{C}_{\min}$ to achieve $\|e\|| \leq \varepsilon$ can be bounded by

$$\mathcal{C}_{\min} \leq \sum_{l=0}^{L_k} |\Delta \mathcal{H}_l(\Gamma)| \cdot J(\tau_0^l, \varepsilon, \overline{\kappa}, L_k, N)$$

whose estimate is given as follows:

Theorem [GWZ14], complexity with hierarchical acceleration

The minimum cost $\mathcal{C}_{\min}$ for building the standard piecewise linear SG approximation $\widetilde{u}_{N_h, M_L}(x, \boldsymbol{y})$ with the prescribed accuracy $\varepsilon > 0$ can be bounded by

$$\mathcal{C}_{\min} \leq \alpha_1 \left[\alpha_2 + \alpha_3 \frac{\log_2\left(\frac{2C_{\text{sg}}}{\varepsilon}\right)}{N} \right]^{\alpha_4 N} \left(\frac{2C_{\text{sg}}}{\varepsilon}\right)^{\alpha_5}$$
$$\times \frac{1}{\log_2\left(\frac{\sqrt{\overline{\kappa}}+1}{\sqrt{\overline{\kappa}}-1}\right)} \left[2N - \log_2(N) + \alpha_9 + \log_2(\sqrt{\overline{\kappa}})\right],$$

where the constants $\alpha_1, \ldots, \alpha_5$ and α_9 are independent of L, N and ε.

Numerical example

Case 1: linear elliptic problem with random inputs

We consider the 2D Poisson equation with random diffusivity and forcing term, i.e.,

$$\begin{cases} \nabla \cdot (a(x, \boldsymbol{y}) \nabla u(x, \boldsymbol{y})) = f(x, \boldsymbol{y}) & \text{in} \quad [0,1]^2 \times \Gamma, \\ u(x, \boldsymbol{y}) = 0 & \text{on} \quad \partial D \times \Gamma, \end{cases}$$

where a and f arethe nonlinear functions of the random vector $\boldsymbol{y}$ given by

$$a(x, \boldsymbol{y}) = 0.1 + \exp\Big[y_1 \cos(\pi x_1) + y_2 \sin(\pi x_2)\Big],$$

and

$$f(x, \boldsymbol{y}) = 10 + \exp\Big[y_3 \cos(\pi x_1) + y_4 \sin(\pi x_2)\Big],$$

where y_n for $n = 1, 2, 3, 4$ are i. i. d. random variables following the uniform distribution $U([-1, 1])$. The quantity of interest is the mean value of the solution over $D \times \Gamma$, i.e.

$$\text{QoI} = \mathbb{E}\left[\int_D u(x, \boldsymbol{y}) dx\right],$$

Numerical example

Case 1: linear elliptic problem with random inputs

The computational savings of the piecewise SG approach with hierarchical acceleration

Basis type	Error	# SG points	hSGSC cost	hSGSC+acceleration cost	hSGSC+acceleration saving
Linear	1.0e-2	377	13,841	7,497	45.8%
	1.0e-3	1,893	81,068	38,670	52.2%
	1.0e-4	7,777	376,287	167,832	55.3%
Quadratic	1.0e-3	701	29,874	11,877	60.2%
	1.0e-4	2,285	110,744	36,760	66.8%
	1.0e-5	6,149	329,294	100,420	69.5%
Cubic	1.0e-4	1,233	59,344	23,228	60.8%
	1.0e-5	3,233	172,845	57,777	66.5%
	1.0e-6	7,079	415,760	129,433	68.8%

Numerical example

Case 2: A stochastic elliptic problem using **glocal** sparse grid SC

Let $\mathbf{x} = (x_1, x_2)$ and consider the following linear elliptic SPDE:

$$\begin{cases} -\nabla \cdot (a(x_1, \boldsymbol{y}(\omega)) \nabla u(\mathbf{x}, \boldsymbol{y}) &= \cos(x_1)\sin(x_2) \quad [0,1]^2 \times \Gamma \\ u(\mathbf{x}, \boldsymbol{y}) &= 0 \quad \text{on } \partial D \times \Gamma \end{cases}$$

The diffusion coefficient is a 1d random field (varies only in x_1) and is $a(x_1, \boldsymbol{y}) = 0.5 + \exp\{\gamma(x_1, \boldsymbol{y})\}$, where γ is a truncated 1d random field with correlation length L and covariance

$$Cov[\gamma](x_1, \tilde{x}_1) = \exp\left(-\frac{(x_1 - \tilde{x}_1)^2}{L^2}\right), \quad \forall (x_1, \tilde{x}_1) \in [0,1]$$

$$\gamma(x_1, \boldsymbol{y}) = 1 + y_1(\omega)\left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} + \sum_{n=2}^{N} \beta_n \, \varphi_n(x_1) \, y_n(\omega)$$

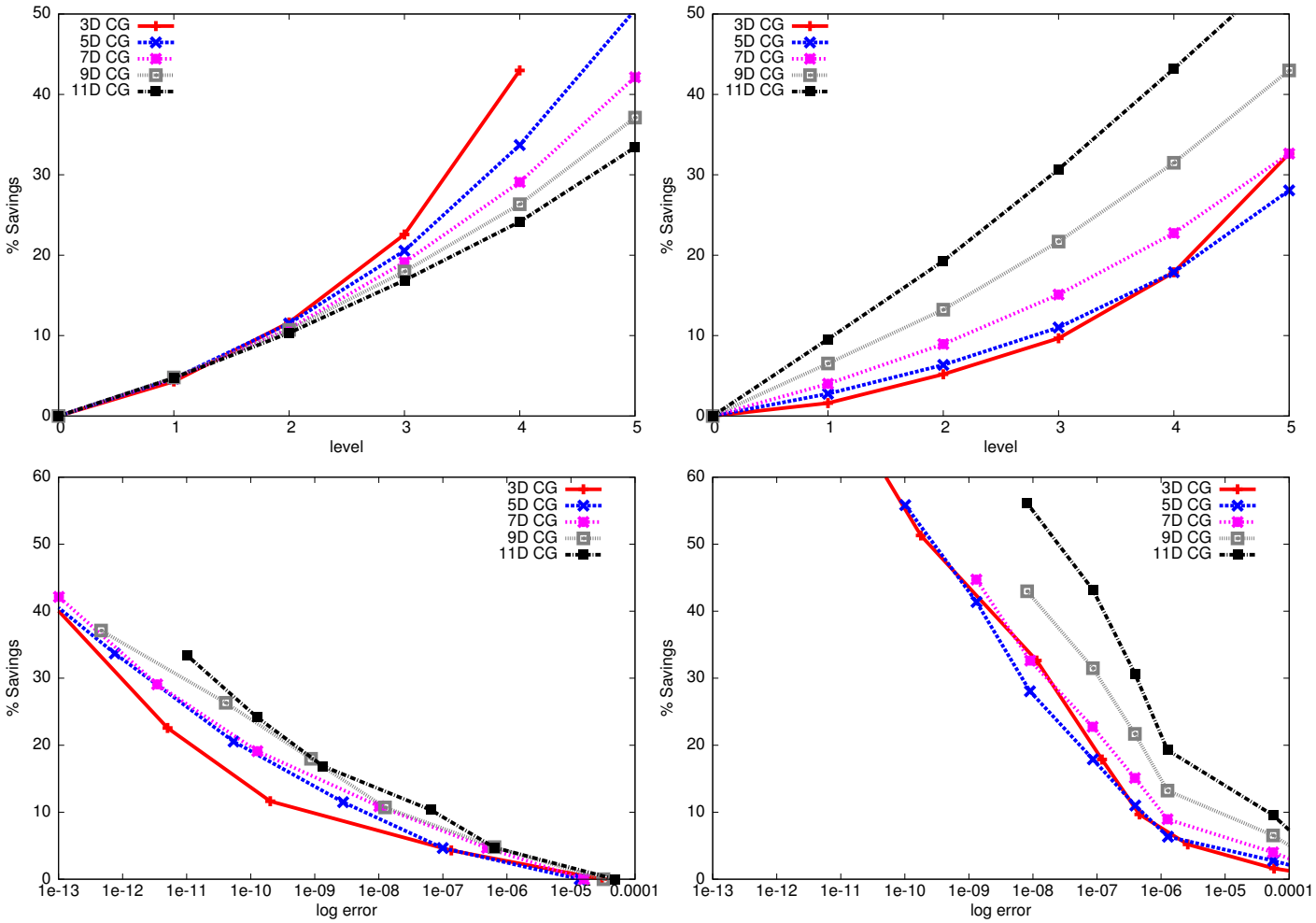
$$\beta_n := \left(\sqrt{\pi}L\right)^{1/2} e^{\frac{-\left(\lfloor \frac{n}{2} \rfloor \pi L\right)^2}{8}}, \quad \varphi_n(x_1) := \begin{cases} \sin\left(\lfloor \frac{n}{2} \rfloor \pi x_1\right), & \text{if } n \text{ even}, \\ \cos\left(\lfloor \frac{n}{2} \rfloor \pi x_1\right), & \text{if } n \text{ odd} \end{cases}$$

- $\mathbb{E}[y_n] = 0$ and $\mathbb{E}[y_n y_m] = \delta_{nm}$ for $n, m \in \mathbb{N}_+$ and iid in $U(-\sqrt{3}, \sqrt{3})$

Computational savings

QoI $= \mathbb{E}[u]$

Savings versus level and savings versus error for $L = 1/64$ (left) and $L = 1/2$ (right)



Conclusion and future work

- The bulk of the computational cost of high-dimensional extreme-scale computational simulations is associated with linear or nonlinear iterative solvers, and the convergence of such methods can be dramatically improved by using the hierarchical acceleration approach
- The new method can be easily extended to other non-intrusive methods, including global sparse-grid approximation, polynomial chaos expansion, etc..
- Extend the new approach to time-dependent problems
- Incorporate this approach into a multi-level framework using model hierarchies $\{u^s(x, \boldsymbol{y})\}_{s=1}^S$ of increasing complexity.

$$\|\widehat{u}^S - u^S\| \leq \|u^0_{M_L} - u^0\| + \sum_{s=1}^{S} \|(u^s_{M_{L-s}} - u^{s-1}_{M_{L-s}}) - (u^s - u^{s-1})\| \leq \epsilon$$

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